§ Motivation

Study moduli problem of Fano type varieties / $\mathbb{C}$

<u>Def</u> X : FT if $\exists$ Δ s.t. ① (X, Δ) is klt

② $-(K_X+\Delta)$ is ample

Similary we can def X/T is of FT.

Let S be a set of varieties. Then S is <u>bounded</u> if $\exists$ $X \to T$ a morphism between schemes of finite type s.t. $\forall$ $Z \in S$, $\exists$ $t \in T$ s.t.

$$Z \simeq X_t \text{ iso.}$$

S is <u>birationally bounded</u> if $\sim \cdots\cdots$ $Z \in S$, $\exists t \in T$

$$Z \sim_{bir} X_t.$$

It is easier to get bir bddness than bddness

(Z smooth Fano, $\exists$ $m = m(d)$ s.t.

$$\phi_{|-mK_Z|} : Z \to \phi(Z) \subseteq \mathbb{P}^N \text{)}$$

$\exists$ Standard strategy to get bddness from bir bddness

Step (I) If $\boxed{Z \underset{iso}{\simeq} X_t}$, $Z' \sim_{bir} Z$,

We want to show $\exists$ a birational map

$$\begin{tikzcd} X \arrow[rr, dashed, "bir"] \arrow[dr] & & X' \arrow[dl] \\ & T & \end{tikzcd} \quad \text{s.t.} \quad X'_t \simeq Z'$$

Step (II) Show there are only finitely many birational models of X/T

$\Rightarrow$ bir bdd + (I) + (II) $\Rightarrow$ bddness

(I) is an extension problem

Suppose $Z \dashrightarrow Z'$ is a seq of mmp, then we want to extend this mmp to $X \dashrightarrow X'$ over T

Note: We can relax T by a finite base change

we can shrink T

(II) is captured by cone of Movable divisors & Nef divisor

[e.g. If X is of FT/T, then both movable / nef cones are polyhedral $\Rightarrow$ we can obtain finiteness)]

This approach are successfuly applied to families of

1. general type varieties
   - Birkar - Casini - Hacon - Mckernan
   - Hacon - Mckernan - Xu
   - Martinelli - Schreider - Tasin
2. Fano varieties
   - Hacon - Xu
   - Birkar
3. Calabi - Yau types
   - Filipazzi - Hacon - Svaldi

Extension Problem (I)

$\boxed{\text{For surface fibration}}$

Extension of (-1) - curves Kodaira 1963

Simultaneous blown down (-1) - curve Iitaka 1970

$\boxed{\text{Deformation of extremal rays / flips}}$

- Kollár - Mori 1992

  (under certain restriction : smooth in codim 2, fibers are $\mathbb{Q}$ - factorial etc.)
- de Fernex - Hacon 2011 , Hacon - Xu 2015

  ~~they~~ they study deformation of various cones of FT varieties
- Shokurov 2020 established more general result in this setting

Rmk : Our result slight generalize [dFH 11] [HX 15]

(II) $\boxed{X \to T}$ surj morph between varieties

$\exists\ S \subseteq T$ X_s are the varieties we want to parametrize.

Conj. If X_s is of FT $\forall\ s \in S$, $\boxed{S \text{ is Zariski dense in } T}$ then X is also of FT over T after shrinking T. and base change

(i.e. if the deformation of a FT variety is still of FT ?)

$\boxed{X \to T}$ $\supseteq S$, $X_s \to S \subseteq T$

$T = \mathbb{C}^1$, $S = \mathbb{Z} \times \mathbb{Z} \subseteq \mathbb{C}^1$

e.g. $\mathbb{F}_m \longrightarrow \mathbb{F}_{m+2}$

~~Ex~~ Donghyeon Kim verified this conjecture for surface fibrations.

We know Cnj holds ① X_s is KLT weak Fano ✓
(X_s KLT, $-K_{X_s}$ is nef + big)

② $\boxed{S = T \setminus \bigcup_{i \in \mathbb{N}} T_i}$, $T_i \subsetneq T$ zariski closed ✓

Difficulty in Cnj : (X_s, Δ_s), $s \in S$

We want to extend $\Delta_s = \Delta|_{X_s}$

We can extend divisor class in $N^1(X/S) = (\mathrm{Pic}(X/S)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$

(I) Setting (✲) ① $X \to T$ ~~smooth~~ fibration
② $S \subseteq T$ zariski dense s.t.

$h^1(X_s, \mathcal{O}_{X_s}) = h^2(X_s, \mathcal{O}_{X_s}) = 0$, X_s has rational sing

$\forall s \in S$.

Rmk. (i) "② is satisfied if X_s are of FT $\forall s \in S$."

(ii) We know $\exists\ U \subseteq T$ open s.t. $X_t,\ t \in U$ satisfies ②

Thm (Kollár-Mori, de Fernex-Hacon, Hacon-Xu, Shokurov, ...)

Under (☆), replacing T by a generic finite base change

$$\boxed{\begin{array}{ccc} N^1(X/T) & \to & N^1(X_t) \\ [D] & \longmapsto & [D|_{X_t}] \end{array}}$$

is an iso $\forall t \in T$.

Pf. $0 \to \mathbb{Z} \to \mathcal{O}_{X_s} \to \mathcal{O}_{X_s}^* \to 0$

$$\Rightarrow \to H^1(\mathcal{O}_{X_s}) \to H^1(\mathcal{O}_{X_s}^*) \xrightarrow{c_1} H^2(X_s, \mathbb{Z}) \to H^2(\mathcal{O}_{X_s}) \to$$

$H^1(\mathcal{O}_{X_s}) = 0$, $H^2(\mathcal{O}_{X_s}) = 0$,

$$\boxed{\mathrm{Pic}(X_s) \cong H^2(X_s, \mathbb{Z})}$$

$\mathrm{Pic}(X_s) = N^1(X_s)$

$\Uparrow$

Topological invariant

By Verdier's result (this is the generalization of Ehresmann's thm in the smooth case), after a Zariski stratification $\boxed{T = \bigcup_{\text{finite}} T_i}$, $X_{T_i} \to T_i$ is locally topologically trivial).

$\Rightarrow\ \forall\ s_1, s_2 \in T_i$, $H^2(X_{s_1}, \mathbb{Z}) \cong H^2(X_{s_2}, \mathbb{Z})$

This identification is up to a monodromy.

In order to "glue" this globally, we need to take a base change to "kill" this monodromy.

Example : $\exists$ fibration $f: X \to T$, $\rho(X/T) = 1$

but $\forall t \in T$, $X_t \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\rho(X_t) = 2$

Fiberwise small $\mathbb{Q}$-factorial modification (SQM)

Thm (-)

Under ($\star$), if X is of klt type, then after a finite base change of T, $\exists$ $Y' \to X/T$ s.t. $Y_t \to X_t$ is a SQM $\forall t \in T$.

(I) Setting ($\star$) ① $X \to T$ fibration

② $S \subseteq T$ Zariski dense s.t.

$h^1(X_s, \mathcal{O}_{X_s}) = h^2(X_s, \mathcal{O}_{X_s}) = 0$, X_s has rational sing

$\forall s \in S$.

Pf $Y' \xrightarrow{g} X/_T$ resolution, run mmp of $\mathrm{Ex}(g)$ over X,

If this mmp is also an mmp of $\mathrm{Ex}(g_t) = \mathrm{Ex}(g)|_{Y'_t}$ over X_t, then the outcome would be SQM of each fiber.

Condition ($\star$) hold for $Y' \to T$, but Y' may NOT be of FT/T

Thm $\Rightarrow$
$$\left.\begin{array}{l} N^1(Y'/T) \simeq N^1(Y'_t) \\ N^1(X/T) \simeq N^1(X_t) \end{array}\right\} \Rightarrow N^1(Y'/X) \simeq N^1(Y'_t/X_t)$$

$\square$

Rmk: ① This fiberwise SQM is crucial in boundedness of FT.

② We don't know if the SQM exists in

genel. a   fiberwise

Thm (de Fernex - Hacon, Hacon - Xu, Shokurov, -)

✓ $\boxed{X/T \text{ is of FT}}$, after a finite base change

$\Rightarrow$ $\mathrm{Eff}(X_t)$, $\mathrm{Nef}(X_t)$ ✓, $\mathrm{Mov}(X_t)$ & Mori chamber decomposition of $\mathrm{Mov}(X_t)$ are all deformation invariant.

① $\underline{\underline{\rho(X_t)}}$ may NOT be constant $\quad X \downarrow T$

$N^1(X_t)$ may NOT deformation invariant.

$h^2 = h^1(\mathcal{O}_{X_t}) = 0 \quad \Leftarrow$

② $\underline{K_{X_t} \equiv 0}$ $\quad U \subset T$, X_U is CY / U.

X_t is of FT $t \in S$, S Zariski dense

$\Rightarrow$ Δ s.t. $K_X + \Delta \sim_{\mathbb{Q}} 0 / T$

(X, Δ) is lc